

A NEW TYPE OF CONTINUED FRACTION EXPANSION

Ion COLTESCU, Dan LASCU

“Mircea cel Batran” Naval Academy, 1 Fulgerului,

900218 Constanta, Romania

e-mail: icoltescu@yahoo.com, lascudan@gmail.com

Abstract

In this paper we define a new type of continued fraction expansion for a real number $x \in I_m := [0, m - 1]$, $m \in N_+$, $m \geq 2$:

$$x = \frac{m^{-b_1(x)}}{1 + \frac{m^{-b_2(x)}}{1 + \ddots}} := [b_1(x), b_2(x), \dots]_m.$$

Then, we derive the basic properties of this continued fraction expansion, following the same steps as in the case of the regular continued fraction expansion. The main purpose of the paper is to prove the convergence of this type of expansion, i.e. we must show that

$$x = \lim_{n \rightarrow \infty} [b_1(x), b_2(x), \dots, b_n(x)]_m.$$

Keywords: *continued fractions, incomplete quotients*

1 INTRODUCTION

In this section we make a brief presentation of the theory of regular continued fraction expansions.

It is well-known that the regular continued fraction expansion of a real number looks as follows:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}} \quad (1)$$

where $a_n \in N$, $\forall n \in N_+$. We can write this expression more compactly as

$$[0; a_1, a_2, \dots, a_n, \dots].$$

The terms a_1, a_2, \dots are called the incomplete quotients of the continued fraction. Continued fractions theory starts with the procedure known as Euclid's algorithm for finding the greatest common divisor. To generalize Euclid's algorithm to irrational numbers from the unit interval I , consider the continued fraction transformation $\tau : I \rightarrow I$ defined by

$$\tau(x) := \frac{1}{x} - \left[\frac{1}{x} \right], x \neq 0, \tau(0) := 0, \quad (2)$$

where $[\cdot]$ denotes the floor (entire) function. Thus, we define $a_1 = a_1(x) = [\frac{1}{x}]$ and $a_n = a_1(\tau^{n-1}(x))$, $\forall n \in N$, where $\tau^0(x) = x$, and $\tau^n(x) = \tau(\tau^{n-1}(x))$. Then, from relation (2), we have:

$$x = \frac{1}{a_1 + \tau(x)} = \frac{1}{a_1 + \frac{1}{a_2 + \tau^2(x)}} = \dots = [0; a_1, a_2, \dots, a_n + \tau^n(x)].$$

The metrical theory of continued fractions expansions is about the sequence $(a_n)_{n \in N}$ of its incomplete quotients, and related sequences. This theory started with Gauss' problem. In modern notation, Gauss wrote that

$$\lim_{n \rightarrow \infty} \lambda(\{x \in [0, 1); \tau^n(x) \leq z\}) = \frac{\log(z+1)}{\log 2}, \quad 0 \leq z \leq 1, \quad (3)$$

where λ is the Lebesgue measure. Gauss asked Laplace to prove (3) and to estimate the error-term $r_n(z)$, defined by $r_n(z) := \lambda(\tau^{-n}([0, z])) - \frac{\log(z+1)}{\log 2}$, $n \geq 1$. (Note that, when we omit the logarithm base, we will consider the natural logarithm.) The first one who proved (3) and at the same time answered Gauss' question was Kuzmin (1928), followed by Lévy. From that time on, a great number of such Gauss-Kuzmin theorems followed. To mention a few: F. Schweiger (1968), P. Wirsing (1974), K.I. Babenko (1978), and more recently by M. Iosifescu (1992).

Apart from regular continued fractions, there are many other continued fractions expansions: Engel continued fractions, Rosen expansions, the nearest integer continued fraction, the grotesque continued fractions, etc.

2 ANOTHER CONTINUED FRACTION EXPANSION

We start this section by showing that any $x \in I_m := [0, m-1]$, $m \in N_+$, $m \geq 2$, can be written in the form

$$\frac{m^{-b_1(x)}}{1 + \frac{m^{-b_2(x)}}{1 + \dots}} := [b_1(x), b_2(x), \dots]_m \quad (4)$$

where $b_n = b_n(x)$ are integer values, belonging to the set $Z_{\geq -1} := \{-1, 0, 1, 2, \dots\}$, for any $n \in N_+$.

Proposition 2.1 For any $x \in I_m := [0, m-1]$, there exist integers numbers $b_n(x) \in \{-1, 0, 1, 2, \dots\}$ such that

$$x = \frac{m^{-b_1(x)}}{1 + \frac{m^{-b_2(x)}}{1 + \ddots}} \quad (5)$$

Proof. If $x \in [0, m-1]$, then we can find an integer $b_1(x) \in Z_{\geq -1}$ such that

$$\frac{1}{m^{b_1(x)+1}} < x < \frac{1}{m^{b_1(x)}}. \quad (6)$$

Thus, there is a $p \in [0, 1]$ such that

$$x = (1-p) \frac{1}{m^{b_1(x)}} + p \frac{1}{m^{b_1(x)+1}} = \frac{m-mp+p}{m} m^{-b_1(x)}.$$

If we set $x_1 = \frac{mp-m}{m-mp+p}$, then we can write x as

$$x = \frac{m^{-b_1(x)}}{1+x_1}.$$

Since $x_1 \in [0, m-1]$, we can repeat the same iteration and obtain

$$x = \frac{m^{-b_1(x)}}{1 + \frac{m^{-b_2(x)}}{1 + \ddots}}$$

which completes the proof.

Next, we define on $I_m := [0, m-1]$, $m \in N_+$, $m \geq 2$, the transformation τ_m by

$$\begin{aligned} \tau_m : I_m &\rightarrow I_m, \\ \tau_m(x) &:= m^{\frac{\log x^{-1}}{\log m} - \left\lfloor \frac{\log x^{-1}}{\log m} \right\rfloor} - 1, x \neq 0, \tau(0) := 0, \end{aligned} \quad (7)$$

where $\lfloor \cdot \rfloor$ denotes the floor (entire) function.

For any $x \in I_m$, put

$$b_n(x) = b_1(\tau_m^{n-1}(x)), n \in N_+,$$

$$b_1(x) = \left\lfloor \frac{\log x^{-1}}{\log m} \right\rfloor, x \neq 0, b_1(0) = \infty.$$

Let Ω_m be the set of all irrational numbers from I_m . In the case when $x \in I_m \setminus \Omega_m$, we have

$$b_n(x) = \infty, \forall n \geq k(x) \geq m, \text{ and } b_n(x) \in Z_{\geq -1}, \forall n < k(x).$$

Therefore, in the rational case, the continued fraction expansion (4) is finite, unlike the irrational case, when we have an infinite number of incomplete quotients from the set $\{-1, 0, 1, 2, \dots\}$.

Let $\omega \in \Omega_m$. We have

$$\omega = m^{\log_m \omega} = m^{-\frac{\log \omega^{-1}}{\log m}} = \frac{m^{-\left[\frac{\log \omega^{-1}}{\log m}\right]}}{m^{\frac{\log \omega^{-1}}{\log m} - \left[\frac{\log \omega^{-1}}{\log m}\right]}} = \frac{m^{-b_1(\omega)}}{1 + \tau_m(\omega)}.$$

Since,

$$\begin{aligned} \tau_m(\omega) &= m^{\log_m \tau_m(\omega)} = m^{-\frac{\log \tau_m^{-1}(\omega)}{\log m}} = \frac{m^{-\left[\frac{\log \tau_m^{-1}(\omega)}{\log m}\right]}}{m^{\frac{\log \tau_m^{-1}(\omega)}{\log m} - \left[\frac{\log \tau_m^{-1}(\omega)}{\log m}\right]}} \\ &= \frac{m^{-b_1(\tau_m(\omega))}}{1 + \tau_m(\tau_m(\omega))} = \frac{m^{-b_2(\omega)}}{1 + \tau_m^2(\omega)} \end{aligned}$$

then, we have

$$\omega = \frac{m^{-b_1(\omega)}}{1 + \frac{m^{-b_2(\omega)}}{1 + \tau_m^2(\omega)}} = \dots = \frac{m^{-b_1(\omega)}}{1 + \frac{m^{-b_2(\omega)}}{1 + \dots + \frac{m^{-b_n(\omega)}}{1 + \tau_m^n(\omega)}}} \quad (8)$$

If $[b_1(\omega)] = m^{-b_1(\omega)}$, and $[b_1(\omega), b_2(\omega), \dots, b_n(\omega)] = \frac{m^{-b_1(\omega)}}{1 + [b_2(\omega), b_3(\omega), \dots, b_n(\omega)]}$, $\forall n \geq 2$, then (8) can be written as

$$\begin{aligned} \omega &= \left[b_1(\omega) + \frac{\log(1 + \tau_m(\omega))}{\log m} \right] = \left[b_1(\omega), b_2(\omega) + \frac{\log(1 + \tau_m^2(\omega))}{\log m} \right] = \dots = \\ &= \left[b_1(\omega), b_2(\omega), \dots, b_{n-1}(\omega), b_n(\omega) + \frac{\log(1 + \tau_m^n(\omega))}{\log m} \right]. \end{aligned}$$

It is obvious that we have the relations

$$\tau_m(\omega) = \frac{m^{-b_2(\omega)}}{1 + \tau_m^2(\omega)}, \dots, \tau_m^{n-1}(\omega) = \frac{m^{-b_n(\omega)}}{1 + \tau_m^n(\omega)}, \forall n \in N_+, \forall \omega \in \Omega_m, \quad (9)$$

3 CONVERGENTS. BASIC PROPERTIES

In this section we define and give the basic properties of the convergents of this continued fraction expansion.

Definition 3.1 A finite truncation in (4), i.e.

$$\omega_n(\omega) := \frac{p_n(\omega)}{q_n(\omega)} = [b_1(\omega), b_2(\omega), \dots, b_n(\omega)]_m, n \in N_+ \quad (10)$$

is called the n -th convergent of ω .

The integer valued functions sequences $(p_n)_{n \in N}$ and $(q_n)_{n \in N}$ can be recursively defined by the formulae:

$$\begin{aligned} p_n(\omega) &= m^{b_n(\omega)} p_{n-1}(\omega) + m^{b_{n-1}(\omega)} p_{n-2}, \forall n \geq 2, \\ q_n(\omega) &= m^{b_n(\omega)} q_{n-1}(\omega) + m^{b_{n-1}(\omega)} q_{n-2}, \forall n \geq 2, \end{aligned} \quad (11)$$

with $p_0(\omega) = 0$, $q_0(\omega) = 1$, $p_1(\omega) = 1$ and $q_1(\omega) = m^{b_1(\omega)}$.

By induction, it is easy to prove that

$$p_n(\omega) q_{n+1}(\omega) - p_{n+1}(\omega) q_n(\omega) = (-1)^{n+1} m^{b_1(\omega) + \dots + b_n(\omega)}, \forall n \in N_+, \quad (12)$$

and that

$$\frac{m^{-b_1(\omega)}}{1 + \frac{m^{-b_2(\omega)}}{1 + \dots + \frac{m^{-b_n(\omega)}}{1+t}}} = \frac{p_n(\omega) + t m^{b_n(\omega)} p_{n-1}(\omega)}{q_n(\omega) + t m^{b_n(\omega)} q_{n-1}(\omega)}, \forall n \in N_+, t \geq 0. \quad (13)$$

Now, combining the relations (8) and (13), it can be shown that, for any $\omega \in \Omega_m$, we have

$$\omega = \frac{p_n(\omega) + \tau_m^n(\omega) m^{b_n(\omega)} p_{n-1}(\omega)}{q_n(\omega) + \tau_m^n(\omega) m^{b_n(\omega)} q_{n-1}(\omega)}, \forall n \in N_+. \quad (14)$$

4 MAIN RESULT

At this moment, we are able to present the main result of the paper, which is the convergence of this new continued fraction expansion, i.e. we must show that

$$\omega = \lim_{n \rightarrow \infty} [b_1(\omega), b_2(\omega), \dots, b_n(\omega)]_m,$$

for any $\omega \in \Omega_m$.

Theorem For any $\omega \in \Omega_m := I_m \setminus Q$, we have

$$\omega - \omega_n(\omega) = \frac{(-1)^n \tau_m^n(\omega) m^{b_1(\omega) + \dots + b_n(\omega)}}{q_n(\omega) (q_n(\omega) + \tau_m^n(\omega) m^{b_n(\omega)} q_{n-1}(\omega))}, \forall n \in N_+. \quad (15)$$

For any $\omega \in \Omega_m$, we have

$$\begin{aligned} \frac{m^{b_1(\omega) + \dots + b_n(\omega)}}{q_n(\omega) (q_{n+1}(\omega) + (m-1)^{n+1} m^{b_{n+1}(\omega)} q_n(\omega))} &< |\omega - \omega_n(\omega)| < \\ &< \frac{1}{\max(F_n, m^{b_1(\omega) + \dots + b_n(\omega)})}, \forall n \in N_+, \end{aligned} \quad (16)$$

and

$$\lim_{n \rightarrow \infty} \omega_n(\omega) = \omega \quad (17)$$

Here F_n denotes the n -th Fibonacci number.

Proof. Using relations (12) and (14), we obtain:

$$\begin{aligned}\omega - \omega_n(\omega) &= \frac{p_n(\omega) + \tau_m^n(\omega)m^{b_n(\omega)}p_{n-1}(\omega)}{q_n(\omega) + \tau_m^n(\omega)m^{b_n(\omega)}q_{n-1}(\omega)} - \frac{p_n(\omega)}{q_n(\omega)} \\ &= \frac{(-1)^n \tau_m^n(\omega)m^{b_1(\omega)+\dots+b_n(\omega)}}{q_n(\omega) (q_n(\omega) + \tau_m^n(\omega)m^{b_n(\omega)}q_{n-1}(\omega))}.\end{aligned}$$

Next, by (9) and (15), it follows:

$$\begin{aligned}|\omega - \omega_n(\omega)| &= \frac{\tau_m^n(\omega)m^{b_1(\omega)+\dots+b_n(\omega)}}{q_n(\omega) (q_n(\omega) + \tau_m^n(\omega)m^{b_n(\omega)}q_{n-1}(\omega))} \\ &= \frac{m^{-b_{n+1}(\omega)}}{1 + \tau_m^{n+1}(\omega)} \cdot \frac{m^{b_1(\omega)+\dots+b_n(\omega)}}{q_n(\omega) \left(q_n(\omega) + \frac{m^{-b_{n+1}(\omega)}}{1 + \tau_m^{n+1}(\omega)} m^{b_n(\omega)}q_{n-1}(\omega) \right)} \\ &= \frac{m^{b_1(\omega)+\dots+b_n(\omega)}}{m^{b_{n+1}(\omega)}q_n(\omega) (q_n(\omega) + \tau_m^{n+1}(\omega)q_n(\omega) + m^{-b_{n+1}(\omega)}m^{b_n(\omega)}q_{n+1}(\omega))} \\ &= \frac{m^{b_1(\omega)+\dots+b_n(\omega)}}{q_n(\omega) (q_{n+1}(\omega) + \tau_m^{n+1}(\omega)m^{b_{n+1}(\omega)}q_n(\omega))} \quad (18)\end{aligned}$$

Now, we know that the Fibonacci numbers are defined by recurrence

$$F_{n+1} = F_n + F_{n-1}, \forall n \in N_+, \text{ and } F_0 = F_1 = 1.$$

Also, from the recurrence relation (11), we infer that

$$p_{n+1} \geq F_{n+1} \text{ and } q_n \geq F_n, \forall n \in N_+, n \geq 2. \quad (19)$$

Also, we have that

$$\begin{aligned}q_n(\omega) &= m^{b_n(\omega)}q_{n-1}(\omega) + m^{b_{n-1}(\omega)}q_{n-2}(\omega) \geq m^{b_n(\omega)}q_{n-1}(\omega) \geq \\ &\geq m^{b_n(\omega)}m^{b_{n-1}(\omega)}q_{n-2}(\omega) \geq \dots \geq m^{b_1(\omega)+\dots+b_n(\omega)}q_0(\omega).\end{aligned}$$

i.e.

$$q_n(\omega) \geq m^{b_1(\omega)+\dots+b_n(\omega)}, \forall n \in N_+. \quad (20)$$

Thus, from relations (19) and (20), we have that

$$q_n(\omega) \geq \max(F_n, m^{b_1(\omega)+\dots+b_n(\omega)}), \forall n \in N_+.$$

Now, since the transformation τ_m belonging to $(0, m-1)$ and from the last two relations, we can show that

$$\begin{aligned}\frac{m^{b_1(\omega)+\dots+b_n(\omega)}}{q_n(\omega) (q_{n+1}(\omega) + \tau_m^{n+1}(\omega)m^{b_{n+1}(\omega)}q_n(\omega))} &\leq \frac{m^{b_1(\omega)+\dots+b_n(\omega)}}{q_n(\omega)q_{n+1}(\omega)} \leq \\ &\leq \frac{1}{q_n(\omega)} \leq \frac{1}{\max(F_n, m^{b_1(\omega)+\dots+b_n(\omega)})}.\end{aligned}$$

It is obvious that the left inequality is true. Since $\max(F_n, m^{b_1(\omega)+\dots+b_n(\omega)})$ is an increasing function, we have

$$\lim_{n \rightarrow \infty} \omega_n(\omega) = \omega.$$

The proof is complete.

5 REMARK

This paper is the first one which addresses this type of continued fraction expansion, and will be followed by other papers which will present the metrical theory of this expansion, the principal aim being solving Gauss' problem.

References

- [1] K. Dajani, C. Kraaikamp, *Ergodic theory of numbers*, Cambridge University Press, 2002.
- [2] A.I. Hincin, *Fractii continue*, Editura Tehnica, Bucuresti, 1960.
- [3] M. Iosifescu, *A very simple proof of a generalization of the Gauss-Kuzmin-Lévy theorem on continued fractions, and related questions*, Rev. Roumaine Math. Pures Appl. 37 (1992), 901-914.
- [4] M. Iosifescu, C. Kraaikamp, *Metrical theory of continued fractions*, Kluwer Academic, 2002.
- [5] M. Iosifescu, G.I. Sebe, *An exact convergence rate in a Gauss-Kuzmin-Lévy problem for some continued fraction expansion*, in vol. Mathematical Analysis and Applications, 90-109. AIP Conf. Proc. 835 (2006), Amer. Inst. Physics, Melville, NY.
- [6] A.M. Rockett, P. Szűsz, *Continued fractions*, World Scientific, Singapore, 1992.
- [7] P. Szűsz, *Über einen Kusminschen Satz*, Acta Math. Acad. Sci. Hungar, 12 (1961), 447-453.
- [8] G.I. Sebe, *A Wirsing-type approach to some continued fraction expansion*, Int. J. Math. Math. Sci., 12 (2005), 1943-1950.
- [9] E. Wirsing, *On the theorem of Gauss-Kuzmin-Lévy and Frobenius-type theorem for function space*, Acta Arithmetica 24 (1974), 507-528.